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Note on a Fixed-Point Theorem of F. E. Browder

NGUYEN HUU ANH

Department of Mathematics,
University of Saigon, Saigon, Viet-Nam

Submitted by Ky Fan

In this note we point to a slight but useful generalization of the following fixed point theorem.

THEOREM 1 (F. E. Browder [1]). *Let f be a completely continuous self-mapping of the Banach space X, m a positive integer. Suppose $f^m(X)$ is bounded. Then f has a fixed point.*

We shall show that the above theorem remains valid if X is replaced by a closed convex subset of X. We need a preliminary lemma.

LEMMA 1. *Let f be a completely continuous mapping of a closed subset Y of a metric space X with distance d into a closed convex subset K of a Banach space E. Then f can be extended to a completely continuous mapping of X into K.*

PROOF. Let x_0 be any point of Y. Let

$$B_n = \{x \in X \mid d(x, x_0) \leqslant n\}.$$

Since f is completely continuous, $\overline{f(B_n \cap Y)}$ is for each positive integer n a compact subset of K. By a theorem of Dugundji [2], the restriction f_1 of f to $B_1 \cap Y$ can be extended to a continuous mapping g_1 of B_1 into a compact subset of K. Let f_2 be defined as follows:

$$f_2(x) = \begin{cases} f(x) & x \in B_2 \cap Y \\ g_1(x) & x \in B_1 \end{cases}$$

f_2 is well defined on $(B_2 \cap Y) \cup B_1 \equiv C$ since $g_1(x) = f(x)$ for

$$x \in B_2 \cap Y \cap B_1 = B_1 \cap Y.$$

Since the restrictions of f_2 on $B_2 \cap Y$ and on B_1 are continuous and since these sets are closed and their union is C, it is clear that f_2 is continuous on C. Again by Dugundji's theorem, f_2 can be extended to a continuous mapping

g_2 of B_2 into a compact subset of K. And so by induction, we define a sequence (g_n) where g_n is a continuous mapping of B_n into a compact subset of K such that g_n agrees with g_{n-1} on B_{n-1}. Let

$$g(x) = g_n(x) \qquad x \in B_n\,, \qquad n = 1, 2, \ldots .$$

Then g is well defined on X and is a completely continuous extension of f. This proves the lemma.

THEOREM 2. *Let f be a completely continuous self-mapping of a closed convex subset K of a Banach space X, m a positive integer. Suppose $f^m(K)$ is bounded. Then f has a fixed point.*

PROOF. By Lemma 1, there exists a completely continuous extension F of f to X with values in K. Then $F^{m+1}(X)$ is bounded. Hence by Theorem 1, F has a fixed point u. Since $u \in K$, we have $f(u) = u$. This proves the theorem.

REFERENCES

1. F. E. BROWDER. On a generalization of the Schauder fixed point theorem. *Duke Math. J.* **26** (1959), 291-303.
2. J. DUGUNDJI. An extension of Tietze's theorem. *Pacific J. Math.* **1** (1951), 353-367.